

# The Art of Linear Algebra

– Graphic Notes on “Linear Algebra for Everyone” –

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with the kindest help of Gilbert Strang †

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## Abstract

I tried intuitive visualizations of important concepts introduced in “Linear Algebra for Everyone”.<sup>1</sup> This is aimed at promoting understanding of vector/matrix calculations and algorithms from the perspectives of matrix factorizations. They include Column-Row ( $CR$ ), Gaussian Elimination ( $LU$ ), Gram-Schmidt Orthogonalization ( $QR$ ), Eigenvalues and Diagonalization ( $Q\Lambda Q^T$ ), and Singular Value Decomposition ( $U\Sigma V^T$ ).

## Foreword

I am happy to see Kenji Hiranabe’s pictures of matrix operations in linear algebra ! The pictures are an excellent way to show the algebra. We can think of matrix multiplications by row  $\cdot$  column dot products, but that is not all – it is “linear combinations” and “rank 1 matrices” that complete the algebra and the art. I am very grateful to see the books in Japanese translation and the ideas in Kenji’s pictures.

– Gilbert Strang  
Professor of Mathematics at MIT

## Contents

<b>1</b>	<b>Viewing a Matrix – 4 Ways</b>	<b>2</b>
<b>2</b>	<b>Vector times Vector – 2 Ways</b>	<b>2</b>
<b>3</b>	<b>Matrix times Vector – 2 Ways</b>	<b>3</b>
<b>4</b>	<b>Matrix times Matrix – 4 Ways</b>	<b>4</b>
<b>5</b>	<b>Practical Patterns</b>	<b>5</b>
<b>6</b>	<b>The Five Factorizations of a Matrix</b>	<b>8</b>
6.1	$A = CR$ . . . . .	9
6.2	$A = LU$ . . . . .	10
6.3	$A = QR$ . . . . .	11
6.4	$S = Q\Lambda Q^T$ . . . . .	12
6.5	$A = U\Sigma V^T$ . . . . .	13

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<sup>1</sup>“Linear Algebra for Everyone”: <http://math.mit.edu/everyone/> with Japanese translation started by Kindai Kagaku.

# 1 Viewing a Matrix – 4 Ways

A matrix ( $m \times n$ ) can be seen as 1 matrix,  $mn$  numbers,  $n$  columns and  $m$  rows.

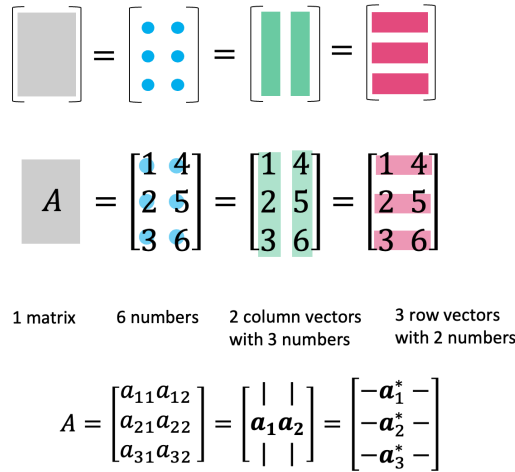


Figure 1: Viewing a Matrix in 4 Ways

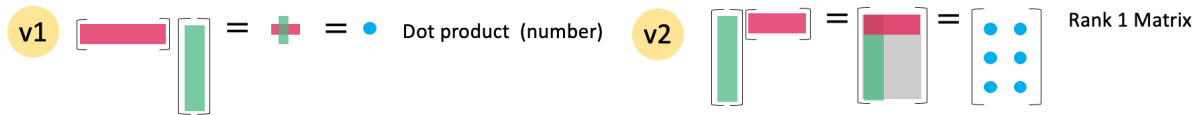
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} | & | \\ \mathbf{a_1} & \mathbf{a_2} \\ | & | \end{bmatrix} = \begin{bmatrix} -\mathbf{a_1}^* - \\ -\mathbf{a_2}^* - \\ -\mathbf{a_3}^* - \end{bmatrix}$$

Here, the column vectors are in bold as  $\mathbf{a_1}$ . Row vectors include  $*$  as in  $\mathbf{a_1}^*$ . Transposed vectors and matrices are indicated by T as in  $\mathbf{a}^T$  and  $A^T$ .

# 2 Vector times Vector – 2 Ways

Hereafter I point to specific sections of “Linear Algebra for Everyone” and present graphics which illustrate the concepts with short names in colored circles.

- Sec. 1.1 (p.2) Linear combination and dot products
- Sec. 1.3 (p.25) Matrix of Rank One
- Sec. 1.4 (p.29) Row way and column way



Dot product ( $a \cdot b$ ) is expressed as  $a^T b$  in matrix language and yields a number.

$ab^T$  is a matrix ( $ab^T = A$ ). If neither  $a, b$  are 0, the result  $A$  is a rank 1 matrix.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 + 2x_2 + 3x_3$$

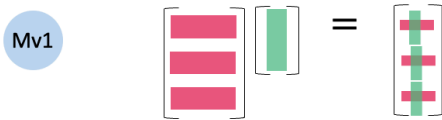
$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} x & y \\ 2x & 2y \\ 3x & 3y \end{bmatrix}$$

Figure 2: Vector times Vector - (v1), (v2)

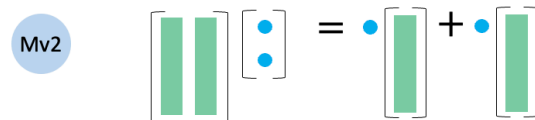
### 3 Matrix times Vector – 2 Ways

A matrix times a vector creates a vector of three dot products (Mv1) as well as a linear combination (Mv2) of the column vectors of  $A$ .

- Sec. 1.1 (p.3) Linear combinations
- Sec. 1.3 (p.21) Matrices and Column Spaces



The row vectors of  $A$  are multiplied by a vector  $\mathbf{x}$  and become the three dot-product elements of  $A\mathbf{x}$ .



The product  $A\mathbf{x}$  is a linear combination of the column vectors of  $A$ .

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (x_1 + 2x_2) \\ (3x_1 + 4x_2) \\ (5x_1 + 6x_2) \end{bmatrix}$$

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Figure 3: Matrix times Vector - (Mv1), (Mv2)

At first, you learn (Mv1). But when you get used to viewing it as (Mv2), you can understand  $A\mathbf{x}$  as a linear combination of the columns of  $A$ . Those products fill the column space of  $A$  denoted as  $\mathbf{C}(A)$ . The solution space of  $A\mathbf{x} = 0$  is the nullspace of  $A$  denoted as  $\mathbf{N}(A)$ .

The four subspaces consists of  $\mathbf{N}(A) + \mathbf{C}(A^T)$  (which are perpendicular to each other) in  $\mathbb{R}^n$  and  $\mathbf{N}(A^T) + \mathbf{C}(A)$  in  $\mathbb{R}^m$  (which are perpendicular to each other).

- Sec. 3.5 (p.124) Dimention of the Four Subspaces

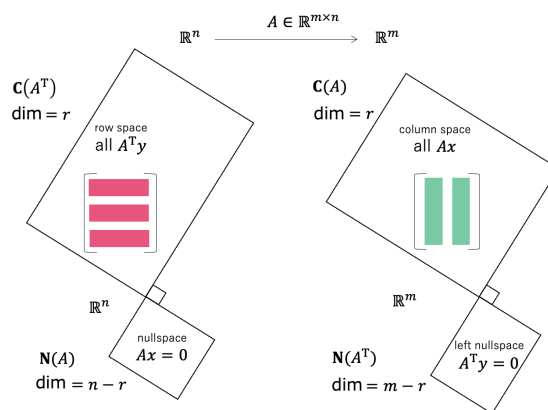


Figure 4: The Four Subspaces

See  $A = CR$  (Sec 6.1) for the rank  $r$ .

## 4 Matrix times Matrix – 4 Ways

“Matrix times Vector” naturally extends to “Matrix times Matrix”.

- Sec. 1.4 (p.35) Four Ways to Multiply  $AB = C$
- Also see the back cover of the book

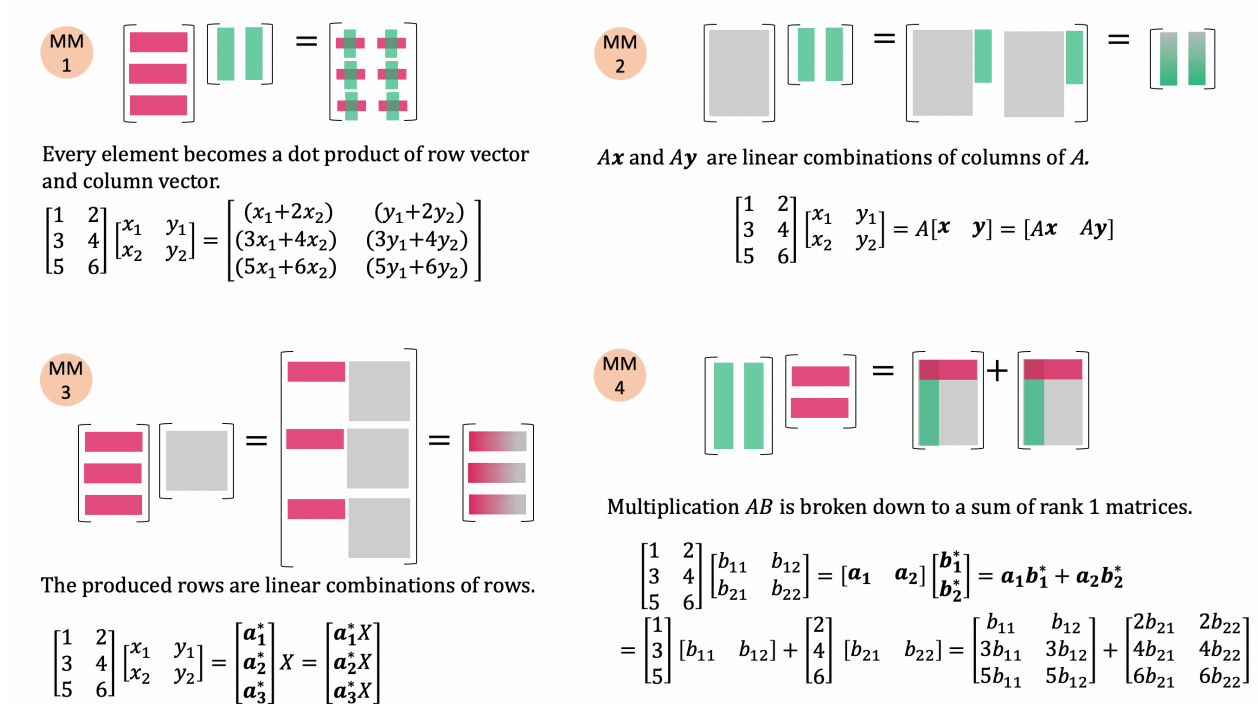


Figure 5: Matrix times Matrix - (MM1), (MM2), (MM3), (MM4)

## 5 Practical Patterns

Here, I show some practical patterns which allow you to capture the coming factorizations more intuitively.

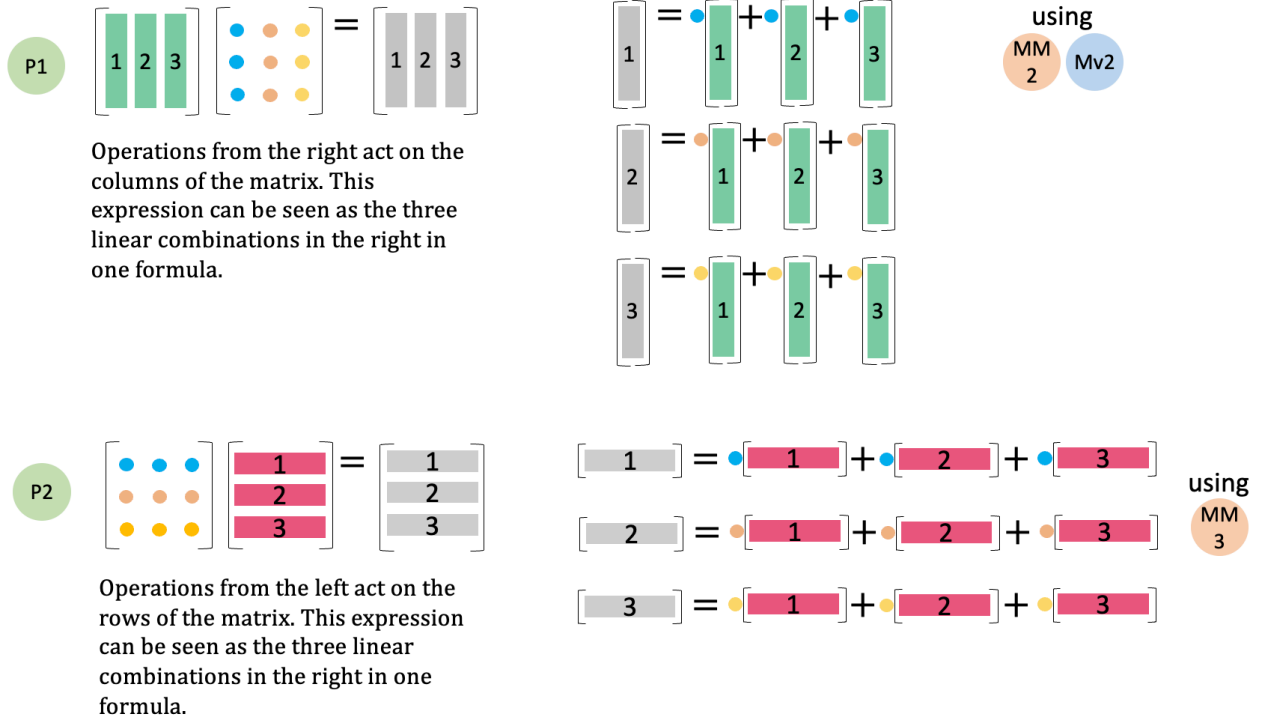
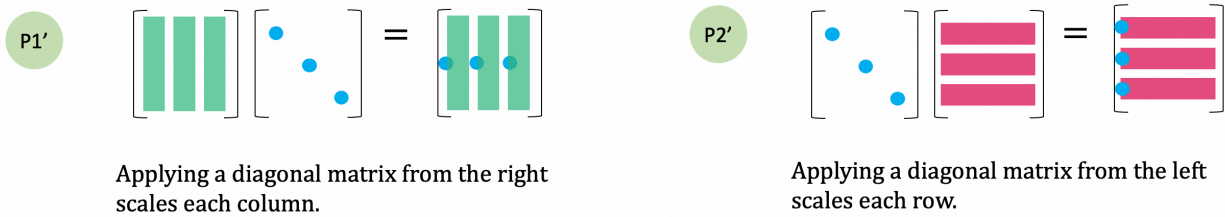


Figure 6: Pattern 1, 2 - (P1), (P1)

Pattern 1 is a combination of (MM2) and (Mv2). Pattern 2 is an extension of (MM3). Note that Pattern 1 is a column operation (multiplying a matrix from right), whereas Pattern 2 is a row operation (multiplying a matrix from left).



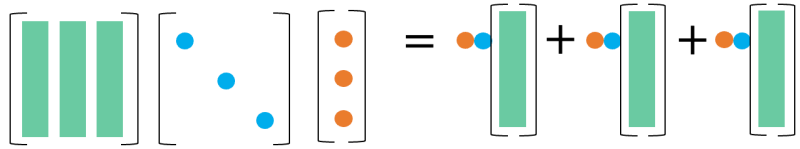
$$AD = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} = [d_1 \mathbf{a}_1 \quad d_2 \mathbf{a}_2 \quad d_3 \mathbf{a}_3]$$

$$DB = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \begin{bmatrix} \mathbf{b}_1^* \\ \mathbf{b}_2^* \\ \mathbf{b}_3^* \end{bmatrix} = \begin{bmatrix} d_1 \mathbf{b}_1^* \\ d_2 \mathbf{b}_2^* \\ d_3 \mathbf{b}_3^* \end{bmatrix}$$

Figure 7: Pattern 1', 2' - (P1'), (P2')

(P1') multiplies the diagonal numbers to the columns of the matrix, whereas (P2') multiplies the diagonal numbers to the row of the matrix. Both are variants of (P1) and (P2).

P3



This pattern makes another combination of columns.  
 You will encounter this in differential/recurrence equations.

$$X D c = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_1 d_1 \mathbf{x}_1 + c_2 d_2 \mathbf{x}_2 + c_3 d_3 \mathbf{x}_3$$

Figure 8: Pattern 3 - (P3)

This pattern appears when you solve differential equations and recurrence equations:

- Sec. 6 (p.201) Eigenvalues and Eigenvectors
- Sec. 6.4 (p.243) Systems of Differential Equations

$$\frac{d\mathbf{u}(t)}{dt} = A\mathbf{u}(t), \quad \mathbf{u}(0) = \mathbf{u}_0$$

$$\mathbf{u}_{n+1} = A\mathbf{u}_n, \quad \mathbf{u}_0 = \mathbf{u}_0$$

In both cases, the solutions are expressed with eigenvalues  $(\lambda_1, \lambda_2, \lambda_3)$ , eigenvectors  $X = [\mathbf{x}_1 \ \mathbf{x}_2 \ \mathbf{x}_3]$  of  $A$ , and the coefficients  $c = [c_1 \ c_2 \ c_3]^T$  which are the coordinates of the initial condition  $\mathbf{u}(0) = \mathbf{u}_0$  in terms of the eigenvectors  $X$ .

$$\mathbf{u}_0 = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + c_3 \mathbf{x}_3$$

$$c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = X^{-1} \mathbf{u}_0$$

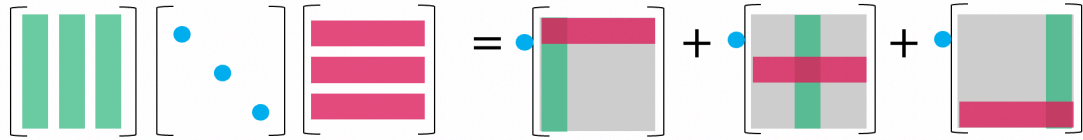
and the general solution of the two equations are:

$$\mathbf{u}(t) = e^{At} \mathbf{u}_0 = X e^{\Lambda t} X^{-1} \mathbf{u}_0 = X e^{\Lambda t} c = c_1 e^{\lambda_1 t} \mathbf{x}_1 + c_2 e^{\lambda_2 t} \mathbf{x}_2 + c_3 e^{\lambda_3 t} \mathbf{x}_3$$

$$\mathbf{u}_n = A^n \mathbf{u}_0 = X \Lambda^n X^{-1} \mathbf{u}_0 = X \Lambda^n c = c_1 \lambda_1^n \mathbf{x}_1 + c_2 \lambda_2^n \mathbf{x}_2 + c_3 \lambda_3^n \mathbf{x}_3$$

See Figure 8: Pattern 3 (P3) above again for  $X D c$ .

P4



A matrix is broken down to a sum of rank 1 matrices, as in singular value/eigenvalue decomposition.

$$U\Sigma V^T = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \mathbf{v}_3^T \end{bmatrix} = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \sigma_3 \mathbf{u}_3 \mathbf{v}_3^T$$

Figure 9: Pattern 4 - (P4)

This pattern (P4) works in both eigenvalue decomposition and singular value decomposition. Both decompositions are expressed as a product of three matrices with a diagonal matrix in the middle, and also a sum of rank 1 matrices with the eigenvalue/singular value coefficients.

More details are discussed in the next section.

## 6 The Five Factorizations of a Matrix

- Preface p.vii, The Plan for the Book.

$A = CR$ ,  $A = LU$ ,  $A = QR$ ,  $A = Q\Lambda Q^T$ ,  $A = U\Sigma V^T$  are illustrated one by one.

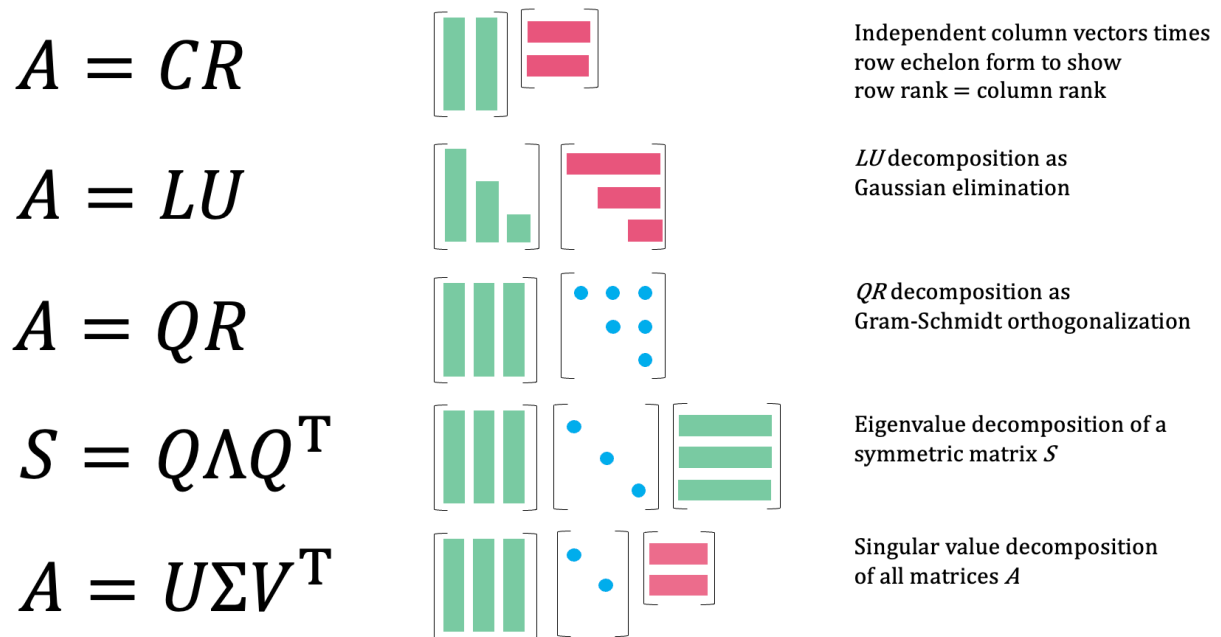


Figure 10: The Five Factorizations



## 6.1 $A = CR$

- Sec.1.4 Matrix Multiplication and  $A = CR$  (p.29)

All general rectangular matrices  $A$  have the same row rank as the column rank. This factorization is the most intuitive way to understand this theorem.  $C$  consists of independent columns of  $A$ , and  $R$  is the row reduced echelon form of  $A$ .  $A = CR$  reduces to  $r$  independent columns in  $C$  times  $r$  independent rows in  $R$ .

$$A = CR$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Procedure: Look at the columns of  $A$  from left to right. Keep independent ones, discard dependent ones which can be created by the former columns. The column 1 and the column 2 survive, and the column 3 is discarded because it is expressed as a sum of the former two columns. To rebuild  $A$  by the independent columns 1, 2, you find a row echelon form  $R$  appearing in the right.

Figure 11: Column Rank in  $CR$

Now you see the column rank is two because there are only two independent columns in  $C$  and all the columns of  $A$  are linear combinations of the two columns of  $C$ .

Figure 12: Row Rank in  $CR$

And you see the row rank is two because there are only two independent rows in  $R$  and all the rows of  $A$  are linear combinations of the two rows of  $R$ .

## 6.2 $A = LU$

Solving  $Ax = b$  via Gaussian elimination can be expressed as a  $LU$  factorization. Usually, you apply elementary row operation matrices ( $E$ ) from left of  $A$  to make upper triangular  $U$ .

$$\begin{aligned} EA &= U \\ A &= E^{-1}U \\ \text{let } L &= E^{-1}, \quad A = LU \end{aligned}$$

Now solve  $Ax = b$  in 2 steps: 1) forward  $Lc = b$  and 2) back  $Ux = c$ .

- Sec.2.3 (p.57) Matrix Computations and  $A = LU$

Here, we directly calculate  $L$  and  $U$  from  $A$ .

$$A = \begin{bmatrix} | \\ | \\ | \\ | \end{bmatrix} [-\mathbf{u}_1^* -] + \begin{bmatrix} 0 & 0 & 0 \\ 0 & & \\ 0 & & A_2 \end{bmatrix} = \begin{bmatrix} | \\ | \\ | \\ | \end{bmatrix} [-\mathbf{u}_1^* -] + \begin{bmatrix} | \\ | \\ | \\ | \end{bmatrix} [-\mathbf{u}_2^* -] + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_3 \end{bmatrix} = LU$$

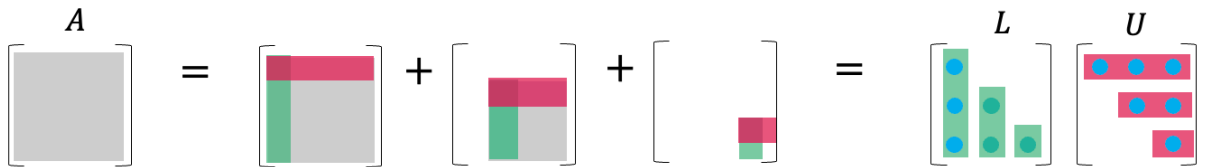


Figure 13: Recursive Rank 1 Matrix Peeling from  $A$

To find  $L$  and  $U$ , peel off the rank 1 matrix made of the first row and the first column of  $A$ . This leaves  $A_2$ . Do this recursively and decompose  $A$  into the sum of rank 1 matrices.

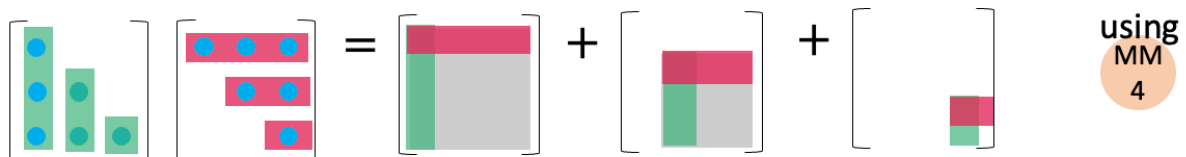


Figure 14:  $LU$  rebuilds  $A$

To rebuild  $A$  from  $L$  times  $U$  is easy.

### 6.3 $A = QR$

$A = QR$  changes the columns of  $A$  into perpendicular columns of  $Q$ , keeping  $C(A) = C(Q)$ .

- Sec.4.4 Orthogonal matrices and Gram-Schmidt (p.165)

In Gram-Schmidt, the normalized  $\mathbf{a}_1$  is picked up as  $\mathbf{q}_1$  first and then  $\mathbf{a}_2$  is adjusted to be perpendicular to  $\mathbf{q}_1$  to create  $\mathbf{q}_2$ , and this procedure goes on.

$$\begin{aligned} \mathbf{q}_1 &= \mathbf{a}_1/|\mathbf{a}_1| \\ \mathbf{q}_2 &= \mathbf{a}_2 - (\mathbf{q}_1^T \mathbf{a}_2)\mathbf{q}_1, \quad \mathbf{q}_2 = \mathbf{q}_2/|\mathbf{q}_2| \\ \mathbf{q}_3 &= \mathbf{a}_3 - (\mathbf{q}_1^T \mathbf{a}_3)\mathbf{q}_1 - (\mathbf{q}_2^T \mathbf{a}_3)\mathbf{q}_2, \quad \mathbf{q}_3 = \mathbf{q}_3/|\mathbf{q}_3| \end{aligned}$$

or you can write:

$$\begin{aligned} \mathbf{a}_1 &= r_{11}\mathbf{q}_1 \\ \mathbf{a}_2 &= r_{12}\mathbf{q}_1 + r_{22}\mathbf{q}_2 \\ \mathbf{a}_3 &= r_{13}\mathbf{q}_1 + r_{23}\mathbf{q}_2 + r_{33}\mathbf{q}_3 \end{aligned}$$

The original  $A$  becomes  $QR$ : orthogonal times triangular.

$$A = \begin{bmatrix} | & | & | \\ \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ & r_{22} & r_{23} \\ & & r_{33} \end{bmatrix} = QR$$

$$QQ^T = Q^TQ = I$$

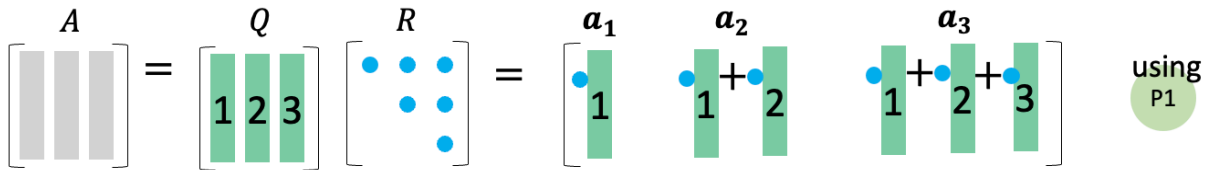


Figure 15:  $A = QR$

The column vectors of  $A$  can be adjusted into an orthonormal set: the column vectors of  $Q$ . Each column vector of  $A$  can be rebuilt from  $Q$  and the upper triangular matrix  $R$ .

See Pattern 1 (P1) again for the graphic interpretation.

## 6.4 $S = Q\Lambda Q^T$

All symmetric matrices  $S$  must have real eigenvalues and orthogonal eigenvectors. The eigenvalues are the diagonal elements of  $\Lambda$  and the eigenvectors are in  $Q$ .

- Sec.6.3 (p.227) Symmetric Positive Definite Matrices

$$\begin{aligned}
 S = Q\Lambda Q^T &= \begin{bmatrix} | & | & | \\ \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \begin{bmatrix} -\mathbf{q}_1^T - \\ -\mathbf{q}_2^T - \\ -\mathbf{q}_3^T - \end{bmatrix} \\
 &= \lambda_1 \begin{bmatrix} | \\ \mathbf{q}_1 \\ | \end{bmatrix} [-\mathbf{q}_1^T -] + \lambda_2 \begin{bmatrix} | \\ \mathbf{q}_2 \\ | \end{bmatrix} [-\mathbf{q}_2^T -] + \lambda_3 \begin{bmatrix} | \\ \mathbf{q}_3 \\ | \end{bmatrix} [-\mathbf{q}_3^T -] \\
 &= \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3
 \end{aligned}$$

$$P_1 = \mathbf{q}_1 \mathbf{q}_1^T, \quad P_2 = \mathbf{q}_2 \mathbf{q}_2^T, \quad P_3 = \mathbf{q}_3 \mathbf{q}_3^T$$

$$\begin{bmatrix} S \end{bmatrix} = \begin{bmatrix} Q \\ \begin{matrix} 1 & 2 & 3 \end{matrix} \end{bmatrix} \begin{bmatrix} \Lambda \\ \begin{matrix} \bullet & & \\ & \bullet & \\ & & \bullet \end{matrix} \end{bmatrix} \begin{bmatrix} Q^T \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \end{bmatrix} = \begin{bmatrix} \lambda_1 \mathbf{q}_1 \mathbf{q}_1^T \\ \begin{matrix} 1 & & \\ & \text{gray} & \end{matrix} \end{bmatrix} + \begin{bmatrix} \lambda_2 \mathbf{q}_2 \mathbf{q}_2^T \\ \begin{matrix} 2 & & \\ & \text{gray} & \end{matrix} \end{bmatrix} + \begin{bmatrix} \lambda_3 \mathbf{q}_3 \mathbf{q}_3^T \\ \begin{matrix} 3 & & \\ & \text{gray} & \end{matrix} \end{bmatrix} \quad \text{using P4}$$

Figure 16:  $S = Q\Lambda Q^T$

A symmetric matrix  $S$  is diagonalized into  $\Lambda$  by an orthogonal matrix  $Q$  and its transpose. And it is broken down into a combination of rank 1 projection matrices  $P = qq^T$ . This is the spectral theorem.

Note that Pattern 4 (P4) is working for the decomposition.

$$\begin{aligned}
 S &= S^T = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 \\
 QQ^T &= P_1 + P_2 + P_3 = I \\
 P_1 P_2 &= P_2 P_3 = P_3 P_1 = O \\
 P_1^2 &= P_1 = P_1^T, \quad P_2^2 = P_2 = P_2^T, \quad P_3^2 = P_3 = P_3^T
 \end{aligned}$$

## 6.5 $A = U\Sigma V^T$

All matrices including rectangular ones have a singular value decomposition (SVD).

$A = U\Sigma V^T$  has the singular vectors of  $A$  in  $U$  and  $V$ . And its singular values in  $\Lambda$ .

- Sec.7.1 (p.259) Singular Values and Singular Vectors

Figure 17:  $A = U\Sigma V^T$

You can find  $V$  as an orthonormal basis of  $\mathbb{R}^n$  (eigenvectors of  $A^T A$ ), and  $U$  as an orthonormal basis of  $\mathbb{R}^m$  (eigenvectors of  $AA^T$ ). Together they diagonalize  $A$  into  $\Sigma$ . This is also expressed as a combination of rank 1 matrices.

$$\begin{aligned}
 A = U\Sigma V^T &= \begin{bmatrix} | & | & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \end{bmatrix} \begin{bmatrix} -\mathbf{v}_1^T - \\ -\mathbf{v}_2^T - \\ \end{bmatrix} = \sigma_1 \begin{bmatrix} | \\ \mathbf{u}_1 \\ | \end{bmatrix} \begin{bmatrix} -\mathbf{v}_1^T - \end{bmatrix} + \sigma_2 \begin{bmatrix} | \\ \mathbf{u}_2 \\ | \end{bmatrix} \begin{bmatrix} -\mathbf{v}_2^T - \end{bmatrix} \\
 &= \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T
 \end{aligned}$$

Note that:

$$\begin{aligned}
 UU^T &= I_m \\
 VV^T &= I_n
 \end{aligned}$$

See Pattern 4 (P4) for the graphic notation.

## Conclusion and Acknowledgements

I presented systematic visualizations of matrix/vector multiplication and their application to the Five Matrix Factorizations. I hope you enjoyed them and will use them in your understanding of Linear Algebra.

Ashley Fernandes helped me with beautifying this paper in typesetting and made it much more consistent and professional.

To conclude this paper, I'd like to thank Prof. Gilbert Strang for publishing "Linear Algebra for Everyone". It guides us through a new vision to these beautiful landscapes in Linear Algebra. Everyone can reach a fundamental understanding of its underlying ideas in a practical manner that introduces us to contemporary and also traditional data science and machine learning. An important part of the matrix world.